補足：ゲージパラメータを固定しない場合

ファインマンゲージで１ループの計算をしましたが、ゲージを固定しない場合だとどうなるのか計算しておきます。この場合何が変わるのかというと、光子の伝播関数は

$$D_{\mu\nu}(k) = -\frac{i}{k^2}\left(g_{\mu\nu} + (\alpha-1)\frac{k_\mu k_\nu}{k^2}\right)$$

これなので、$(\alpha-1)k_\mu k_\nu/k^4$ という項が引っ付いてきます。なので、１ループではフェルミオンの自己エネルギーと頂点補正にはこの項による影響がでてきます。というわけで、この項を同じように次元正則化を使って計算していきます (面倒なので、発散項だけを取り出します)。

- 自己エネルギー

  １ループでの自己エネルギーは

$$-i\Sigma(p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{i}{\not{p}-\not{k}-m}\gamma_\nu D^{\mu\nu}(k)$$

  このような形なので、計算すべきなのは

$$\begin{aligned}
-i\Sigma_G(p) &= (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{i}{\not{p}-\not{k}-m}\gamma_\nu\frac{-ik^\mu k^\nu}{k^4}\\
&= -e^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{1}{\not{p}-\not{k}-m}\gamma_\nu\frac{k^\mu k^\nu}{k^4}\\
&= -e^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu\frac{\not{p}-\not{k}+m}{(p-k)^2-m}\gamma_\nu\frac{k^\mu k^\nu}{k^4}
\end{aligned}$$

  次元を d 次元にして

$$-i\Sigma_G(p) = -e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\gamma_\mu\frac{\not{p}-\not{k}+m}{(p-k)^2-m}\gamma_\nu\frac{k^\mu k^\nu}{k^4}$$

  ここからは、ガンマ行列の計算、ファインマンパラメータによる置き換え、次元正則化を行っていきます。分子は

$$\begin{aligned}
\gamma_\mu(\not{p}-\not{k}+m)\gamma_\nu k^\mu k^\nu &= \not{k}\not{p}\not{k} - \not{k}\not{k}\not{k} + m\not{k}\not{k}\\
&= \not{k}p_\mu\gamma^\mu\gamma^\nu k_\nu - k^2\not{k} + mk^2\\
&= \not{k}p_\mu(2g^{\mu\nu}-\gamma^\nu\gamma^\mu)k_\nu - k^2\not{k} + mk^2\\
&= 2\not{k}p\cdot k - k^2\not{p} - k^2\not{k} + mk^2\\
&= -k^2(\not{p}+\not{k}-m) + 2\not{k}p\cdot k
\end{aligned}$$

  と計算されるので

$$\begin{aligned}
-i\Sigma_G(p) &= -e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{-k^2(\not{p}+\not{k}-m)+2\not{k}p\cdot k}{k^4(p-k)^2-m} \\
&= -e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\Big[-\frac{\not{p}+\not{k}-m}{k^2(p-k)^2-m}+\frac{2\not{k}p\cdot k}{k^4(p-k)^2-m}\Big] \\
&= I_1+I_2
\end{aligned}$$

使うファインマンパラメータは

$$\frac{1}{AB}=\int_0^1 dx\frac{1}{(xA+(1-x)B)^2}\ ,\ \frac{1}{AB^2}=\int_0^1 dx\frac{2(1-x)}{(xA+(1-x)B)^3}$$

この二つです。ファインマンパラメータによる置き換えはファインマンゲージでの場合と同じなので、分母の形は I_1, I_2 両方とも

$$k'^2-m^2x+p^2x(1-x)\ \ (k'=k-px)$$

となります。というわけで、I_1 は

$$\begin{aligned}
I_1 &= -e^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{-(\not{p}+\not{k}-m)}{[(p-k)^2-m^2]k^2} \\
&= -e^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk'}{(2\pi)^d}\frac{-(\not{p}+\not{k}'+\not{x}p-m)}{[k'^2-m^2x+p^2x(1-x)]^2} \\
&= -e^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{-\not{p}(1+x)+m}{[k^2-m^2x+p^2x(1-x)]^2}
\end{aligned}$$

I_2 は

$$\begin{aligned}
I_2 &= -e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{2\not{k}p\cdot k}{[(p-k)^2-m^2](k^2)^2} \\
&= -e^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{2\not{k}p\cdot k}{[x\{(p-k)^2-m^2\}+(1-x)k^2]^3} \\
&= -2e^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk'}{(2\pi)^d}\frac{2(1-x)(\not{k}'+x\not{p})p\cdot(k'+px)}{[k'^2-m^2x+p^2x(1-x)]^3} \\
&= -2e^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk'}{(2\pi)^d}\frac{2(1-x)A}{[k'^2-m^2x+p^2x(1-x)]^3}
\end{aligned}$$

$$\begin{aligned}
A &= (\not{k}'+x\not{p})p\cdot(k'+px) \\
&= \not{k}'(p\cdot k')+\not{k}'p^2x+x\not{p}p\cdot k'+\not{p}p^2x^2 \\
&= (\not{k}'+x\not{p})(p\cdot k')+p^2x(\not{k}'+x\not{p}) \\
&= (\not{k}'(p\cdot k')+x^2p^2\not{p}=A_1+A_2
\end{aligned}$$

k_μ だけを含む項は積分で消えるので消しています。これらから必要な k 積分は

$$T_1 = \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - m^2x + p^2x(1-x)]^2} \ , \ T_2 = \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - m^2x + p^2x(1-x)]^3}$$

$$T_3^{\mu\nu} = \int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2 - m^2x + p^2x(1-x)]^3}$$

この 3 つとなります ($\not{k}(p \cdot k) = k^\nu \gamma_\nu p_\mu k^\mu$)。これらはユークリッド化して積分を行うことで

$$T_1 = \int \frac{id^dk_E}{(2\pi)^d} \frac{1}{[k_E^2 + m^2x - p^2x(1-x)]^2} = \frac{i}{(4\pi)^{d/2}} \Big(\frac{1}{m^2x - p^2x(1-x)} \Big)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}$$

$$\begin{aligned}
T_2 &= -\int \frac{id^dk_E}{(2\pi)^d} \frac{1}{[k_E^2 + m^2x - p^2x(1-x)]^3} = \frac{-i}{(4\pi)^{d/2}} \Big(\frac{1}{m^2x - p^2x(1-x)} \Big)^{3-\frac{d}{2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \\
&= \frac{-i}{(4\pi)^{d/2}} \Big(\frac{1}{m^2x - p^2x(1-x)} \Big)^{\frac{\epsilon}{2}+1} \frac{\Gamma(\frac{\epsilon}{2}+1)}{\Gamma(3)}
\end{aligned}$$

$$\begin{aligned}
T_3^{\mu\nu} &= \frac{g^{\mu\nu}}{d} \int \frac{id^dk_E}{(2\pi)^d} \frac{k_E^2}{[k_E^2 + m^2x - p^2x(1-x)]^3} \\
&= \frac{ig^{\mu\nu}}{(4\pi)^{d/2}} \frac{1}{2} \Big(\frac{1}{m^2x - p^2x(1-x)} \Big)^{3-\frac{d}{2}-1} \frac{\Gamma(3-\frac{d}{2}-1)}{\Gamma(3)} \\
&= \frac{ig^{\mu\nu}}{(4\pi)^{d/2}} \frac{1}{2} \Big(\frac{1}{m^2x - p^2x(1-x)} \Big)^{\frac{\epsilon}{2}} \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(3)}
\end{aligned}$$

これらを I_1, I_2 に入れることで

$$I_1 = -e^2(\mu^2)^{\epsilon/2}\int_0^1 dx\int \frac{d^dk}{(2\pi)^d}\frac{-\not{p}(1+x)+m}{[k^2-m^2x+p^2x(1-x)]^2}$$

$$= -e^2(\mu^2)^{\epsilon/2}\int_0^1 dx(-\not{p}(1+x)+m)T_1$$

$$= \frac{-ie^2(\mu^2)^{\epsilon/2}}{(4\pi)^{-\epsilon/2}(4\pi)^2}\int_0^1 dx(-\not{p}(1+x)+m)\Big(\frac{1}{m^2x-p^2x(1-x)}\Big)^{\frac{\epsilon}{2}}\frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}$$

$$= \frac{-ie^2}{(4\pi)^2\Gamma(2)}\int_0^1 dx(-\not{p}(1+x)+m)\Big(\frac{4\pi\mu^2}{m^2x-p^2x(1-x)}\Big)^{\frac{\epsilon}{2}}\Gamma(\frac{\epsilon}{2})$$

$$= \frac{-ie^2}{(4\pi)^2}\int_0^1 dx(-\not{p}(1+x)+m)\Big(\frac{4\pi\mu^2}{m^2x-p^2x(1-x)}\Big)^{\frac{\epsilon}{2}}\Gamma(\frac{\epsilon}{2})$$

$$I_2 = -2e^2(\mu^2)^{\epsilon/2}\int_0^1 dx\int \frac{d^dk'}{(2\pi)^d}\frac{2(1-x)(A_1+A_2)}{[k'^2-m^2x+p^2x(1-x)]^3}$$

$$= -4e^2(\mu^2)^{\epsilon/2}\int_0^1 dx(1-x)(\gamma_\nu p_\mu T_3^{\mu\nu}+x^2p^2\not{p}T_2)$$

$$= \frac{-4ie^2(\mu^2)^{\epsilon/2}}{(4\pi)^{-\epsilon/2}(4\pi)^2\Gamma(3)}\int_0^1 dx(1-x)\Big[\frac{1}{2}g^{\mu\nu}\gamma_\nu p_\mu\Gamma(\frac{\epsilon}{2})-x^2p^2\not{p}\Big(\frac{1}{m^2x-p^2x(1-x)}\Big)\frac{\epsilon}{2}\Gamma(\frac{\epsilon}{2})\Big]\Big(\frac{1}{m^2x-p^2x(1-x)}\Big)^{\frac{\epsilon}{2}}$$

$$= \frac{-2ie^2}{(4\pi)^2}\int_0^1 dx(1-x)\Big[\frac{1}{2}\not{p}-x^2p^2\not{p}\Big(\frac{1}{m^2x-p^2x(1-x)}\Big)\frac{\epsilon}{2}\Big]\Big(\frac{4\pi\mu^2}{m^2x-p^2x(1-x)}\Big)^{\frac{\epsilon}{2}}\Gamma(\frac{\epsilon}{2})$$

$\epsilon \to 0$ とすることで、$-i\Sigma_G(p) = I_1 + I_2$ は

$$
\begin{aligned}
-i\Sigma_G(p) =&\frac{-ie^2}{(4\pi)^2}\Gamma(\frac{\epsilon}{2})\int_0^1 dx\big(\frac{4\pi\mu^2}{m^2x-p^2x(1-x)}\big)^{\frac{\epsilon}{2}}[(-\not{p}(1+x)+m)\\
&\qquad\qquad\qquad\qquad +2(1-x)\big(\frac{1}{2}\not{p}-\frac{\epsilon}{2}x^2p^2\not{p}\frac{1}{m^2x-p^2x(1-x)}\big)]\\
=&\frac{-ie^2}{(4\pi)^2}(\frac{2}{\epsilon}-\gamma)\int_0^1 dx\big(\frac{4\pi\mu^2}{m^2x-p^2x(1-x)}\big)^{\frac{\epsilon}{2}}[(-\not{p}(1+x)+m)\\
&\qquad\qquad\qquad\qquad +(1-x)\big(\not{p}-\epsilon x^2p^2\not{p}\frac{1}{m^2x-p^2x(1-x)}\big)]\\
=&\frac{-ie^2}{(4\pi)^2}\int_0^1 dx\big(\frac{4\pi\mu^2}{m^2x-p^2x(1-x)}\big)^{\frac{\epsilon}{2}}\big[\big(-\not{p}(1+x)+m\big)(\frac{2}{\epsilon}-\gamma)\\
&\qquad\qquad\qquad\qquad +(1-x)\big(\not{p}(\frac{2}{\epsilon}-\gamma)-(2-\gamma\epsilon)x^2p^2\not{p}\frac{1}{m^2x-p^2x(1-x)}\big)\big]\\
=&\frac{-ie^2}{(4\pi)^2}\int_0^1 dx\big(1-\frac{\epsilon}{2}\log\big[\frac{-p^2x(1-x)+m^2x}{4\pi\mu^2}\big]\big)\big[(-\not{p}(1+x)+m)(\frac{2}{\epsilon}-\gamma)\\
&\qquad\qquad\qquad\qquad +(1-x)\not{p}(\frac{2}{\epsilon}-\gamma-(2-\gamma\epsilon)x^2p^2\frac{1}{m^2x-p^2x(1-x)}\big)\big]\\
=&\frac{-ie^2}{(4\pi)^2}\int_0^1 dx(1-\frac{\epsilon}{2}\log[\frac{-p^2x(1-x)+m^2x}{4\pi\mu^2}])\\
&\qquad\times\big[F\frac{2}{\epsilon}-F\gamma+\not{p}(1-x)\big(\frac{2}{\epsilon}-\gamma-(2-\gamma\epsilon)x^2p^2\frac{1}{m^2x-p^2x(1-x)}\big)\big]\quad (F=-\not{p}(1+x)+m)\\
=&\frac{-ie^2}{(4\pi)^2}\int_0^1 dx\big[\frac{2}{\epsilon}(-\not{p}(1+x)+m)+\frac{2}{\epsilon}(1-x)\not{p}\big]+finite\\
=&\frac{-ie^2}{(4\pi)^2}\frac{2}{\epsilon}\int_0^1 dx[-2x\not{p}+m]+finite\\
=&\frac{-ie^2}{8\pi^2\epsilon}(-\not{p}+m)+finite
\end{aligned}
$$

これが自己エネルギーの場合新しく加わる項です。

- 頂点補正

  頂点補正でも同様に

$$
\begin{aligned}
-ie\Gamma_\mu^G(p,q,p+q) &= (-ie\mu^{2-d/2})^3\int\frac{d^dk}{(2\pi)^d}\gamma_\alpha\frac{i}{\not{p}+\not{q}-\not{k}-m}\gamma_\mu\frac{i}{\not{p}-\not{k}-m}\gamma_\beta\frac{-ik^\alpha k^\beta}{k^4}\\
&= -(e\mu^{2-d/2})^3\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta\gamma_\alpha(\not{p}'-\not{k}+m)\gamma_\mu(\not{p}-\not{k}+m)\gamma_\beta}{(k^2)^2\big((p'-k)^2-m^2\big)\big((p-k)^2-m^2\big)}
\end{aligned}
$$

を計算すればいいことになります $(p'=p+q)$。分子の計算で出てくるガンマ行列を計算すると

$$
\begin{aligned}
k^\alpha k^\beta p^{\rho\prime} p^\lambda \gamma_\alpha\gamma_\rho\gamma_\mu\gamma_\lambda\gamma_\beta &= k^\alpha k^\beta p^{\rho\prime} p^\lambda (2g_{\alpha\rho} - \gamma_\rho\gamma_\alpha)\gamma_\mu\gamma_\lambda\gamma_\beta \\
&= 2k^\alpha k^\beta p^{\rho\prime} p^\lambda g_{\alpha\rho}\gamma_\mu\gamma_\lambda\gamma_\beta - k^\alpha k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\alpha\gamma_\mu\gamma_\lambda\gamma_\beta \\
&= 2k_\rho k^\beta p^{\rho\prime} p^\lambda \gamma_\mu\gamma_\lambda\gamma_\beta - k^\alpha k^\beta p^{\rho\prime} p^\lambda \gamma_\rho(2g_{\alpha\mu} - \gamma_\mu\gamma_\alpha)\gamma_\lambda\gamma_\beta \\
&= 2k_\rho k^\beta p^{\rho\prime} p^\lambda \gamma_\mu\gamma_\lambda\gamma_\beta - 2k_\mu k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\lambda\gamma_\beta + k^\alpha k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu\gamma_\alpha\gamma_\lambda\gamma_\beta \\
&= 2k_\rho k^\beta p^{\rho\prime} p^\lambda \gamma_\mu\gamma_\lambda\gamma_\beta - 2k_\mu k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\lambda\gamma_\beta + k^\alpha k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu(2g_{\alpha\lambda} - \gamma_\lambda\gamma_\alpha)\gamma_\beta \\
&= 2k_\rho k^\beta p^{\rho\prime} p^\lambda \gamma_\mu\gamma_\lambda\gamma_\beta - 2k_\mu k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\lambda\gamma_\beta + 2k_\lambda k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu\gamma_\beta - k^\alpha k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu\gamma_\lambda\gamma_\alpha\gamma_\beta \\
&= 2k_\rho k^\beta p^{\rho\prime} p^\lambda \gamma_\mu\gamma_\lambda\gamma_\beta - 2k_\mu k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\lambda\gamma_\beta + 2k_\lambda k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu\gamma_\beta - k^2 p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu\gamma_\lambda \\
&= 2(k\cdot p')k^\beta p^\lambda \gamma_\mu\gamma_\lambda\gamma_\beta - 2k_\mu k^\beta p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\lambda\gamma_\beta + 2(k\cdot p)k^\beta p^{\rho\prime} \gamma_\rho\gamma_\mu\gamma_\beta - k^2 p^{\rho\prime} p^\lambda \gamma_\rho\gamma_\mu\gamma_\lambda
\end{aligned}
$$

4 個の場合で

$$
\begin{aligned}
k^\alpha k^\beta p^\rho \gamma_\alpha\gamma_\mu\gamma_\rho\gamma_\beta &= k^\alpha k^\beta p^\rho (2g_{\alpha\mu} - \gamma_\mu\gamma_\alpha)\gamma_\rho\gamma_\beta \\
&= k^\alpha k^\beta p^\rho (2g_{\alpha\mu}\gamma_\rho\gamma_\beta - \gamma_\mu\gamma_\alpha\gamma_\rho\gamma_\beta) \\
&= k^\alpha k^\beta p^\rho (2g_{\alpha\mu}\gamma_\rho\gamma_\beta - \gamma_\mu(2g_{\alpha\rho} - \gamma_\rho\gamma_\alpha)\gamma_\beta) \\
&= k^\alpha k^\beta p^\rho (2g_{\alpha\mu}\gamma_\rho\gamma_\beta - 2g_{\alpha\rho}\gamma_\mu\gamma_\beta + \gamma_\mu\gamma_\rho\gamma_\alpha\gamma_\beta) \\
&= 2k_\mu k^\beta p^\rho \gamma_\rho\gamma_\beta - 2(k\cdot p)k^\beta\gamma_\mu\gamma_\beta + k^2 p^\rho\gamma_\mu\gamma_\rho
\end{aligned}
$$

3 個の場合で

$$k^\alpha k^\beta \gamma_\alpha\gamma_\mu\gamma_\beta = k^\alpha k^\beta(2g_{\alpha\mu} - \gamma_\mu\gamma_\alpha)\gamma_\beta = (2k_\mu k^\beta\gamma_\beta - k^2\gamma_\mu)$$

このように計算されるので、分子は

$$
\begin{aligned}
&k^{\alpha}k^{\beta}\gamma_{\alpha}(\not{p}'-\not{k}+m)\gamma_{\mu}(\not{p}-\not{k}+m)\gamma_{\beta}\\
=&k^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\not{p}\gamma_{\beta}-k^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\not{k}\gamma_{\beta}+k^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\gamma_{\beta}m-k^{\alpha}k^{\beta}\gamma_{\alpha}\not{k}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{\alpha}k^{\beta}\gamma_{\alpha}\not{k}\gamma_{\mu}\not{k}\gamma_{\beta}-k^{\alpha}k^{\beta}\gamma_{\alpha}\not{k}\gamma_{\mu}\gamma_{\beta}m\\
&+k^{\alpha}k^{\beta}\gamma_{\alpha}m\gamma_{\mu}\not{p}\gamma_{\beta}-k^{\alpha}k^{\beta}\gamma_{\alpha}m\gamma_{\mu}\not{k}\gamma_{\beta}+m^{2}k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\\
=&k^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\not{p}\gamma_{\beta}-k^{2}k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}+mk^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\gamma_{\beta}-k^{2}k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{4}\gamma_{\mu}-mk^{2}k^{\beta}\gamma_{\mu}\gamma_{\beta}\\
&+mk^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\not{p}\gamma_{\beta}-mk^{2}k^{\alpha}\gamma_{\alpha}\gamma_{\mu}+m^{2}k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\\
=&2(k\cdot p')k^{\beta}p^{\lambda}\gamma_{\mu}\gamma_{\lambda}\gamma_{\beta}-2k_{\mu}k^{\beta}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\lambda}\gamma_{\beta}+2(k\cdot p)k^{\beta}p^{\rho\prime}\gamma_{\rho}\gamma_{\mu}\gamma_{\beta}-k^{2}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\mu}\gamma_{\lambda}\\
&+m(k^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\gamma_{\beta}+k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\not{p}\gamma_{\beta})+m^{2}k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\\
&+k^{2}(-k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}-k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{2}\gamma_{\mu}-mk^{\beta}\gamma_{\mu}\gamma_{\beta}-mk^{\alpha}\gamma_{\alpha}\gamma_{\mu})\\
=&2(k\cdot p')k^{\beta}p^{\lambda}\gamma_{\mu}\gamma_{\lambda}\gamma_{\beta}-2k_{\mu}k^{\beta}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\lambda}\gamma_{\beta}+2(k\cdot p)k^{\beta}p^{\rho\prime}\gamma_{\rho}\gamma_{\mu}\gamma_{\beta}\\
&+m(k^{\alpha}k^{\beta}\gamma_{\alpha}\not{p}'\gamma_{\mu}\gamma_{\beta}+k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\not{p}\gamma_{\beta})+m^{2}k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\\
&+k^{2}(-k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}-k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{2}\gamma_{\mu}-mk^{\beta}\gamma_{\mu}\gamma_{\beta}-mk^{\alpha}\gamma_{\alpha}\gamma_{\mu}-p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\mu}\gamma_{\lambda})\\
=&2(k\cdot p')k^{\beta}p^{\lambda}\gamma_{\mu}\gamma_{\lambda}\gamma_{\beta}-2k_{\mu}k^{\beta}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\lambda}\gamma_{\beta}+2(k\cdot p)k^{\beta}p^{\rho\prime}\gamma_{\rho}\gamma_{\mu}\gamma_{\beta}\\
&+m(2k_{\mu}k^{\beta}p^{\rho}\gamma_{\rho}\gamma_{\beta}-2(k\cdot p)k^{\beta}\gamma_{\mu}\gamma_{\beta}+k^{2}p^{\rho}\gamma_{\mu}\gamma_{\rho}+2(k\cdot p')k^{\beta}\gamma_{\mu}\gamma_{\beta}-2k_{\mu}k^{\beta}p'^{\rho}\gamma_{\rho}\gamma_{\beta}+k^{2}p'^{\rho}\gamma_{\rho}\gamma_{\mu})\\
&+m^{2}k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}+k^{2}(-k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}-k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{2}\gamma_{\mu}-mk^{\beta}\gamma_{\mu}\gamma_{\beta}-mk^{\alpha}\gamma_{\alpha}\gamma_{\mu}-p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\mu}\gamma_{\lambda})\\
=&2(k\cdot p')k^{\beta}p^{\lambda}\gamma_{\mu}\gamma_{\lambda}\gamma_{\beta}-2k_{\mu}k^{\beta}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\lambda}\gamma_{\beta}+2(k\cdot p)k^{\beta}p^{\rho\prime}\gamma_{\rho}\gamma_{\mu}\gamma_{\beta}\\
&+m(2k_{\mu}k^{\beta}p^{\rho}\gamma_{\rho}\gamma_{\beta}-2(k\cdot p)k^{\beta}\gamma_{\mu}\gamma_{\beta}+2(k\cdot p')k^{\beta}\gamma_{\mu}\gamma_{\beta}-2k_{\mu}k^{\beta}p'^{\rho}\gamma_{\rho}\gamma_{\beta})\\
&+k^{2}(-k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}-k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{2}\gamma_{\mu}-mk^{\beta}\gamma_{\mu}\gamma_{\beta}-mk^{\alpha}\gamma_{\alpha}\gamma_{\mu}-p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\mu}\gamma_{\lambda}+mp'^{\rho}\gamma_{\rho}\gamma_{\mu}+mp^{\rho}\gamma_{\mu}\gamma_{\rho})\\
&+m^{2}k^{\alpha}k^{\beta}\gamma_{\alpha}\gamma_{\mu}\gamma_{\beta}\\
=&2(k\cdot p')k^{\beta}p^{\lambda}\gamma_{\mu}\gamma_{\lambda}\gamma_{\beta}-2k_{\mu}k^{\beta}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\lambda}\gamma_{\beta}+2(k\cdot p)k^{\beta}p^{\rho\prime}\gamma_{\rho}\gamma_{\mu}\gamma_{\beta}\\
&+m(2k_{\mu}k^{\beta}p^{\rho}\gamma_{\rho}\gamma_{\beta}-2(k\cdot p)k^{\beta}\gamma_{\mu}\gamma_{\beta}+2(k\cdot p')k^{\beta}\gamma_{\mu}\gamma_{\beta}-2k_{\mu}k^{\beta}p'^{\rho}\gamma_{\rho}\gamma_{\beta})\\
&+k^{2}(-k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}-k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{2}\gamma_{\mu}-mk^{\beta}\gamma_{\mu}\gamma_{\beta}-mk^{\alpha}\gamma_{\alpha}\gamma_{\mu}-p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\mu}\gamma_{\lambda}+mp'^{\rho}\gamma_{\rho}\gamma_{\mu}+mp^{\rho}\gamma_{\mu}\gamma_{\rho})\\
&+m^{2}(2k_{\mu}k^{\beta}\gamma_{\beta}-k^{2}\gamma_{\mu})\\
=&2(k\cdot p')k^{\beta}p^{\lambda}\gamma_{\mu}\gamma_{\lambda}\gamma_{\beta}-2k_{\mu}k^{\beta}p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\lambda}\gamma_{\beta}+2(k\cdot p)k^{\beta}p^{\rho\prime}\gamma_{\rho}\gamma_{\mu}\gamma_{\beta}\\
&+m(2k_{\mu}k^{\beta}p^{\rho}\gamma_{\rho}\gamma_{\beta}-2(k\cdot p)k^{\beta}\gamma_{\mu}\gamma_{\beta}+2(k\cdot p')k^{\beta}\gamma_{\mu}\gamma_{\beta}-2k_{\mu}k^{\beta}p'^{\rho}\gamma_{\rho}\gamma_{\beta})\\
&+k^{2}(-k^{\alpha}\gamma_{\alpha}\not{p}'\gamma_{\mu}-k^{\beta}\gamma_{\mu}\not{p}\gamma_{\beta}+k^{2}\gamma_{\mu}-mk^{\beta}\gamma_{\mu}\gamma_{\beta}-mk^{\alpha}\gamma_{\alpha}\gamma_{\mu}-p^{\rho\prime}p^{\lambda}\gamma_{\rho}\gamma_{\mu}\gamma_{\lambda}+mp'^{\rho}\gamma_{\rho}\gamma_{\mu}+mp^{\rho}\gamma_{\mu}\gamma_{\rho}-m^{2}\gamma_{\mu})\\
&+2m^{2}k_{\mu}k^{\beta}\gamma_{\beta}\\
=&B_{1}+k^{2}B_{2}
\end{aligned}
$$

B_1 に対するファインマンパラメータは

$$\frac{1}{A_1^2A_2A_3}=\int_0^1 dx_2\int_0^{1-x_2}dx_3\frac{6(1-x_2-x_3)}{((1-x_2-x_3)A_1+x_2A_2+x_3A_3)^4}$$

B_2 に対しては

$$\frac{1}{A_1A_2A_3}=\int_0^1 dx_2\int_0^{1-x_2}dx_3\frac{2}{((1-x_2-x_3)A_1+x_2A_2+x_3A_3)^3}$$

どちらの場合でも分母はファインマンゲージの時と同じで

$$k'^2+p'^2x(1-x)+p^2y(1-y)-2p\cdot p'xy-m^2(x+y)\quad(k=k'+p'x+py)$$

このような格好をします。そうすると発散を起こす項というのは、次元正則化の結果である

$$\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+a)^n}=\frac{1}{(4\pi)^{d/2}}\frac{d}{2}\big(\frac{1}{a}\big)^{n-\frac{d}{2}-1}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}$$

$$\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+a)^n}=\frac{1}{(4\pi)^{d/2}}\big(\frac{1}{a}\big)^{n-\frac{d}{2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}$$

この二つから、l_E^2 を含んでおり $n=3$ のときに可能だということが分かります。今の場合分母は 3 か 4 乗になっているので、分母が 3 乗で分子に k^2 がいる項のみが発散します。というわけで、B_2 の中で $k=k'+p'x+py$ の置き換えをした時に、k'^2 を含むものを取り出します

$$\begin{aligned}B_2=&-k^\alpha\gamma_\alpha\not{p}'\gamma_\mu-k^\beta\gamma_\mu\not{p}\gamma_\beta\\&+k^2\gamma_\mu-mk^\beta\gamma_\mu\gamma_\beta-mk^\alpha\gamma_\alpha\gamma_\mu-p^{\rho\prime}p^\lambda\gamma_\rho\gamma_\mu\gamma_\lambda+mp'^\rho\gamma_\rho\gamma_\mu+mp^\rho\gamma_\mu\gamma_\rho-m^2\gamma_\mu\\&\Rightarrow k'^2\gamma_\mu\end{aligned}$$

よって、ファインマンパラメータの置き換えによって

$$\begin{aligned}&-ie\Gamma_\mu^G(p,q,p+q)\\&=-(e\mu^{2-d/2})^3\int_0^1dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{2k'^2\gamma_\mu}{(k'^2+p'^2x(1-x)+p^2y(1-y)-2p\cdot p'xy-m^2(x+y))^3}+finite\end{aligned}$$

これは、ファインマンゲージでの場合と同じ積分なので、同様にして

$$
\begin{aligned}
&-ie\Gamma_\mu^{G(pole)}(p,q,p+q)\\
&\quad= -(e\mu^{2-d/2})^3\int_0^1 dx\int_0^{1-x}dy\int \frac{id^dk_E}{(2\pi)^d}\frac{2k_E^2\gamma_\mu}{(k_E'^2-p'^2x(1-x)-p^2y(1-y)+2p\cdot p'xy+m^2(x+y))^3}\\
&\quad= -2i(e\mu^{2-d/2})^3\gamma_\mu\frac{d}{2}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(3)}\int_0^1 dx\int_0^{1-x}dy(\frac{1}{-p'^2x(1-x)-p^2y(1-y)+2p\cdot p'xy+m^2(x+y)})^{2-\frac{d}{2}}\\
&\quad= -id(e\mu^{2-d/2})^3\frac{\gamma_\mu}{(4\pi)^2\Gamma(3)}\Gamma(\frac{\epsilon}{2})\int_0^1 dx\int_0^{1-x}dy(\frac{4\pi\mu^2}{-p'^2x(1-x)-p^2y(1-y)+2p\cdot p'xy+m^2(x+y)})^{2-\frac{d}{2}}
\end{aligned}
$$

$\epsilon\to 0$ にして、発散項だけを取り出せば

$$
\begin{aligned}
-ie\Gamma_\mu^{G(pole)}(p,q,p+q) &= -i4(e\mu^{2-d/2})^3\frac{\gamma_\mu}{(4\pi)^2 2}\frac{2}{\epsilon}\int_0^1 dx\int_0^{1-x}dy\\
&= -i4(e\mu^{2-d/2})^3\frac{\gamma_\mu}{(4\pi)^2 2}\frac{2}{\epsilon}\frac{1}{2}
\end{aligned}
$$

よって

$$
\Gamma_\mu^{G(pole)}(p,q,p+q)=\frac{e^2}{8\pi^2\epsilon}\gamma_\mu
$$

この二つの結果に $(\alpha-1)$ をかけてファインマンゲージの結果に足せばゲージパラメータを固定していないものが求まります。
今の結果を使うことで、$\alpha=0$ であるランダウゲージでの 1 ループの計算結果は

$$
\begin{aligned}
-i\Sigma &= -i\Sigma_F-(-i\Sigma_G)\\
&= \frac{ie^2}{8\pi^2\epsilon}(\not p-4m)+\frac{ie^2}{8\pi^2\epsilon}(-\not p+m)\\
&= \frac{-3ie^2}{8\pi^2\epsilon}m
\end{aligned}
$$

$$
\Gamma_\mu^{(pole)}=\Gamma_\mu^{F(pole)}-\Gamma_\mu^{G(pole)}=\frac{e^2}{8\pi^2\epsilon}\gamma_\mu-\frac{e^2}{8\pi^2\epsilon}\gamma_\mu=0
$$

これらより、頂点とフェルミオン場に対するくり込みを行う必要性がなくなっていることが分かります。これはワード高橋恒等式を

$$
\begin{aligned}
q_\mu\Gamma^\mu &= S^{-1}(p+q)-S^{-1}(p)\\
\Rightarrow q_\mu\gamma^\mu &= \not p+\not q-m_r-\not p+m_r\\
&= \not q
\end{aligned}
$$

このように成立させていることになります。
実際に計算して分かったように、ファインマンゲージを取った時とランダウゲージを取った時で値が変わっています。これは一般的に n 点グリーン関数はゲージ不変な量ではないということの表れです。そのために、ゲージ不変性はワード・高橋恒等式 (非可換ゲージでは Slavnov-Taylor 恒等式) を通して要求されることになっています。