ガンマ行列の内積とトレース

ガンマ行列の内積とトレースをまとめています。

レヴィ・チビタ記号 $\epsilon^{\mu\nu\alpha\beta}$ は

$$\epsilon^{0123} = -\epsilon_{0123} = +1$$

と定義します。偶置換では $+1$ ($\epsilon^{0123} = \epsilon^{0312} = \cdots = +1$)、奇置換では -1 ($\epsilon^{1023} = \cdots = -1$) です。

ガンマ行列の関係

- $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$
- $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$
- $\gamma^0 = \gamma_0$
- $\gamma^{i\dagger} = \gamma^0\gamma^i\gamma^0 = -\gamma^i$
- $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$
- $\gamma_0 = \gamma_0^\dagger = (\gamma_0)^{-1},\ \gamma^{i\dagger} = (\gamma^i)^{-1}$
- $(\gamma^5)^\dagger = \gamma^5,\ (\gamma^5)^2 = 1,\ \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$
- $\gamma^5 = -\dfrac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$

内積

- $\gamma_\mu\gamma^\mu = 4$
- $\gamma_\mu\gamma^\alpha\gamma^\mu = -2\gamma^\alpha$
- $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4g^{\alpha\beta}$
- $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\mu = -2\gamma^\rho\gamma^\beta\gamma^\alpha$
- $\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\lambda\gamma^\mu = 2\gamma^\lambda\gamma^\alpha\gamma^\beta\gamma^\rho + 2\gamma^\rho\gamma^\beta\gamma^\alpha\gamma^\lambda$
- $A_\mu A_\nu\gamma^\mu\gamma^\nu = A^2$

トレース

- $\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}] = 0$ (n は奇数)
- $\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_{2n}}] = \mathrm{tr}[\gamma^{\alpha_{2n}}\cdots\gamma^{\alpha_2}\gamma^{\alpha_1}]$
- $\mathrm{tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}$
- $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = 4(g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta})$
- $\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}] = \pm 4\sum g^{\alpha_{l_1}\alpha_{m_1}}g^{\alpha_{l_2}\alpha_{m_2}}\cdots g^{\alpha_{l_n}\alpha_{m_n}}$ (n は偶数)
- $\mathrm{tr}\gamma^5 = 0$
- $\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu] = 0$

- $\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = -4i\epsilon^{\mu\nu\alpha\beta}$
- $A_\alpha B_\beta \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta] = 4(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B))$
- $A_\alpha B_\beta \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta] = 8(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B) - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta)$
- $A_\alpha B_\beta C^\rho D^\lambda \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta]\mathrm{tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\lambda] = 32((A\cdot C)(B\cdot D) + (A\cdot D)(B\cdot C))$
- $A_\alpha B_\beta C^\rho D^\lambda \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta]\mathrm{tr}[\gamma_\mu(1-\gamma^5)\gamma_\rho\gamma_\nu(1-\gamma^5)\gamma_\lambda] = 256(A\cdot C)(B\cdot D)$

内積とトレースを示していきます。$\gamma_\mu\gamma^\mu$ は

$$\gamma_\mu\gamma^\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = g_{\mu\nu}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2\delta^\mu_\mu - \gamma_\mu\gamma^\mu = 8 - \gamma_\mu\gamma^\mu$$

から、$\gamma_\mu\gamma^\mu = 4$ です。ガンマ行列が間にあるときは

$$\gamma_\mu\gamma^\alpha\gamma^\mu = \gamma_\mu(2g^{\alpha\mu} - \gamma^\mu\gamma^\alpha) = 2\gamma^\alpha - 4\gamma^\alpha = -2\gamma^\alpha$$

2 個では

$$\begin{aligned}\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = \gamma_\mu\gamma^\alpha(2g^{\mu\beta} - \gamma^\mu\gamma^\beta) = 2\gamma^\beta\gamma^\alpha - \gamma_\mu\gamma^\alpha\gamma^\mu\gamma^\beta &= 2\gamma^\beta\gamma^\alpha + 2\gamma^\alpha\gamma^\beta \\ &= 2(2g^{\alpha\beta} - \gamma^\alpha\gamma^\alpha) + 2\gamma^\alpha\gamma^\beta \\ &= 4g^{\alpha\beta}\end{aligned}$$

3 個では

$$\begin{aligned}\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\mu = \gamma_\mu\gamma^\alpha\gamma^\beta(2g^{\rho\mu} - \gamma^\mu\gamma^\rho) &= 2g^{\rho\mu}\gamma_\mu\gamma^\alpha\gamma^\beta - \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\rho \\ &= 2\gamma^\rho(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha) - 4g^{\alpha\beta}\gamma^\rho \\ &= \ -2\gamma^\rho\gamma^\beta\gamma^\alpha\end{aligned}$$

4 個では

$$\begin{aligned}\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\lambda\gamma^\mu = \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho(2g^{\lambda\mu} - \gamma^\mu\gamma^\lambda) &= 2g^{\lambda\mu}\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho - \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\mu\gamma^\lambda \\ &= 2\gamma^\lambda\gamma^\alpha\gamma^\beta\gamma^\rho + 2\gamma^\rho\gamma^\beta\gamma^\alpha\gamma^\lambda\end{aligned}$$

$A_\mu A_\nu\gamma^\mu\gamma^\nu$ は

$$A_\mu A_\nu\gamma^\mu\gamma^\nu = A_\mu A_\nu(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2A^2 - A_\mu A_\nu\gamma^\nu\gamma^\mu = 2A^2 - A_\nu A_\mu\gamma^\mu\gamma^\nu = 2A^2 - A_\mu A_\nu\gamma^\mu\gamma^\nu$$

なので、$A_\mu A_\nu\gamma^\mu\gamma^\nu = A^2$ です。
　トレースを求めます。

- $\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}]=0$ (n は奇数)

  トレースの巡回性から

$$\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}]=\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\gamma^5\gamma^5]=\mathrm{tr}[\gamma^5\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\gamma^5]$$

  γ^5 は他のガンマ行列と反交換するので

$$\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}]=(-1)^n\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}\gamma^5\gamma^5]=(-1)^n\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}]$$

  よって、n が奇数では 0 です。

- $\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_{2n}}]=\mathrm{tr}[\gamma^{\alpha_{2n}}\cdots\gamma^{\alpha_2}\gamma^{\alpha_1}]$

  「荷電共役」での変換 C は「T」を転置として $C^{-1}\gamma^\mu C=-(\gamma^\mu)^T$ なので

$$\begin{aligned}\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_{2n}}]&=\mathrm{tr}[CC^{-1}\gamma^{\alpha_1}CC^{-1}\gamma^{\alpha_2}CC^{-1}\cdots CC^{-1}\gamma^{\alpha_{2n}}CC^{-1}]\\&=\mathrm{tr}[C^{-1}\gamma^{\alpha_1}CC^{-1}\gamma^{\alpha_2}CC^{-1}\cdots CC^{-1}\gamma^{\alpha_{2n}}C]\\&=\mathrm{tr}[(\gamma^{\alpha_1})^T(\gamma^{\alpha_2})^T\cdots(\gamma^{\alpha_{2n}})^T]\\&=\mathrm{tr}[(\gamma^{\alpha_{2n}}\cdots\gamma^{\alpha_2}\gamma^{\alpha_1})^T]\\&=\mathrm{tr}[\gamma^{\alpha_{2n}}\cdots\gamma^{\alpha_2}\gamma^{\alpha_1}]\qquad(\mathrm{tr}A^T=\mathrm{tr}A)\end{aligned}$$

- $\mathrm{tr}[\gamma^\mu\gamma^\nu]=4g^{\mu\nu}$

  巡回性と $\mathrm{tr}[A+B]=\mathrm{tr}A+\mathrm{tr}B$ から

$$\mathrm{tr}[\gamma^\mu\gamma^\nu]=\frac{1}{2}(\mathrm{tr}[\gamma^\mu\gamma^\nu]+\mathrm{tr}[\gamma^\nu\gamma^\mu])=\frac{1}{2}(\mathrm{tr}[\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu])=g^{\mu\nu}\mathrm{tr}1=4g^{\mu\nu}$$

- $\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta]=4(g^{\mu\nu}g^{\alpha\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\alpha}g^{\nu\beta})$

  γ^μ を一番右側に移動させると

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta=(2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^\alpha\gamma^\beta&=2g^{\mu\nu}\gamma^\alpha\gamma^\beta-\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta\\&=2g^{\mu\nu}\gamma^\alpha\gamma^\beta-\gamma^\nu(2g^{\mu\alpha}-\gamma^\alpha\gamma^\mu)\gamma^\beta\\&=2g^{\mu\nu}\gamma^\alpha\gamma^\beta-2g^{\mu\alpha}\gamma^\nu\gamma^\beta+\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\\&=2g^{\mu\nu}\gamma^\alpha\gamma^\beta-2g^{\mu\alpha}\gamma^\nu\gamma^\beta+\gamma^\nu\gamma^\alpha(2g^{\mu\beta}-\gamma^\beta\gamma^\mu)\\&=2g^{\mu\nu}\gamma^\alpha\gamma^\beta-2g^{\mu\alpha}\gamma^\nu\gamma^\beta+2g^{\mu\beta}\gamma^\nu\gamma^\alpha-\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu\end{aligned}$$

トレースを取ると

$$\begin{aligned}\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] &= 2g^{\mu\nu}\mathrm{tr}[\gamma^\alpha\gamma^\beta] - 2g^{\mu\alpha}\mathrm{tr}[\gamma^\nu\gamma^\beta] + 2g^{\mu\beta}\mathrm{tr}[\gamma^\nu\gamma^\alpha] - \mathrm{tr}[\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu]\\ &= 8g^{\mu\nu}g^{\alpha\beta} - 8g^{\mu\alpha}g^{\nu\beta} + 8g^{\mu\beta}g^{\nu\alpha} - \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta]\\ &= 4g^{\mu\nu}g^{\alpha\beta} - 4g^{\mu\alpha}g^{\nu\beta} + 4g^{\mu\beta}g^{\nu\alpha}\end{aligned}$$

- $\mathrm{tr}[\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n}] = \pm 4\sum g^{\alpha_{l_1}\alpha_{m_1}}g^{\alpha_{l_2}\alpha_{m_{21}}}\cdots g^{\alpha_{l_n}\alpha_{m_n}}$ (n は偶数)
  $l_1 < l_2 < \cdots < l_n,\ l_i < m_i$ として和を取り、符号は $l_1, m_1, l_2, m_2, \ldots, l_n, m_n$ の並びが偶置換で $1, 2, \ldots, n$ になるならプラス、奇置換ならマイナスです。

  $n = 4$ は、計量を $g^{\alpha_1\alpha_2} = g_{(12)}$ ガンマ行列を $\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_n} = \alpha_1\alpha_2\cdots\alpha_n$ と書くことにして

$$\begin{aligned}\mathrm{tr}[\alpha_1\alpha_2\alpha_3\alpha_4] &= 2g_{(12)}\mathrm{tr}[\alpha_3\alpha_4] - 2g_{(13)}\mathrm{tr}[\alpha_2\alpha_4] + 2g_{(14)}\mathrm{tr}[\alpha_2\alpha_3] - \mathrm{tr}[\alpha_2\alpha_3\alpha_4\alpha_1]\\ &= g_{(12)}\mathrm{tr}[\alpha_3\alpha_4] - g_{(13)}\mathrm{tr}[\alpha_2\alpha_4] + g_{(14)}\mathrm{tr}[\alpha_2\alpha_3]\\ &= 4g_{(12)}g_{(34)} - 4g_{(13)}g_{(24)} + 4g_{(14)}g_{(23)}\\ &= 4g_{(12)}g_{(34)} - 4g_{(13)}g_{(24)} + 4g_{(14)}g_{(23)}\end{aligned}$$

$n = 6$ では

$$\begin{aligned}\mathrm{tr}[\alpha_1\cdots\alpha_6] &= 2g_{(12)}\mathrm{tr}[\alpha_3\alpha_4\alpha_5\alpha_6] - 2g_{(13)}\mathrm{tr}[\alpha_2\alpha_4\alpha_5\alpha_6]\\ &\quad + 2g_{(14)}\mathrm{tr}[\alpha_2\alpha_3\alpha_5\alpha_6] - 2g_{(15)}\mathrm{tr}[\alpha_2\alpha_3\alpha_4\alpha_6] + 2g_{(16)}\mathrm{tr}[\alpha_2\alpha_3\alpha_4\alpha_5]\\ &\quad - \mathrm{tr}[\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_1]\\ &= g_{(12)}\mathrm{tr}[\alpha_3\alpha_4\alpha_5\alpha_6] - g_{(13)}\mathrm{tr}[\alpha_2\alpha_4\alpha_5\alpha_6] + g_{(14)}\mathrm{tr}[\alpha_2\alpha_3\alpha_5\alpha_6]\\ &\quad - g_{(15)}\mathrm{tr}[\alpha_2\alpha_3\alpha_4\alpha_6] + g_{(16)}\mathrm{tr}[\alpha_2\alpha_3\alpha_4\alpha_5]\end{aligned}$$

第 1 項は

$$\begin{aligned}g_{(12)}\mathrm{tr}[\alpha_3\alpha_4\alpha_5\alpha_6] &= g_{(12)}(4g_{(34)}g_{(56)} - 4g_{(35)}g_{(46)} + 4g_{(36)}g_{(45)})\\ &= 4g_{(12)}g_{(34)}g_{(56)} - 4g_{(12)}g_{(35)}g_{(46)} + 4g_{(12)}g_{(36)}g_{(45)}\end{aligned}$$

第 2 項は

$$\begin{aligned}-g_{(13)}\mathrm{tr}[\alpha_2\alpha_4\alpha_5\alpha_6] &= -g_{(13)}(4g_{(24)}g_{(56)} - 4g_{(25)}g_{(46)} + 4g_{(26)}g_{(45)})\\ &= -4g_{(13)}g_{(24)}g_{(56)} + 4g_{(13)}g_{(25)}g_{(46)} - 4g_{(13)}g_{(26)}g_{(45)}\end{aligned}$$

これで法則性が分かります。

$n=4$ のとき、計量の添え字の並びと符号は

$$+(1,2)(3,4)\ ,\ -(1,3)(2,4)\ ,\ +(1,4)(2,3)$$

これは $1,2,3,4$ の並びに対して偶置換ならプラス、奇置換ならマイナスです。そして、添え字 $(i,j)(k,l)$ の i に対して $i<j,\ i<k$ です。$n=6$ でも同様で

$$\begin{aligned}&+(1,2)(3,4)(5,6)\ ,\ -(1,2)(3,5)(4,6)\ ,\ +(1,2)(3,6)(4,5)\\&-(1,3)(2,4)(5,6)\ ,\ +(1,3)(2,5)(4,6)\ ,\ -(1,3)(2,6)(4,6)\end{aligned}$$

というわけで、$l_1<l_2<\cdots<l_n,\ l_i<m_i$ として和を取ることで

$$\mathrm{tr}[\alpha_1\alpha_2\cdots\alpha_n]=\pm4\sum g(\alpha_{l_1}\alpha_{m_1})g(\alpha_{l_2}\alpha_{m_2})\cdots g(\alpha_{l_n}\alpha_{m_n})$$

$g^{\alpha_{l_i}\alpha_{m_i}}$ を $g(\alpha_{l_i}\alpha_{m_i})$ と書いています。符号は $l_1,m_1,l_2,m_2,\dots,l_n,m_n$ の並びが偶置換で $1,2,\dots,n$ の並びになるならプラス、奇置換ならマイナスです。

ちなみに $\mathrm{tr}[\alpha_1\alpha_2\cdots\alpha_n]$ で出てくる項の数は $(2n)!/(2^nn!)$ です。$n=6$ の計算を見れば分かるように、α_1 を一番右まで交換していくとき $n-1$ 個の項が現れ、その各項にいる $\mathrm{tr}[\alpha_1\alpha_2\cdots\alpha_{n-2}]$ から $(n-2)-1$ 個の項が現れ、というように続きます。$n=6$ では、まず $6-1=5$ 個の項が現れ、各項でさらに現れる $4-1=3$ 個の項との積を取って、15 個です。というわけで、偶数なので $\mathrm{tr}(\alpha_1\alpha_2\cdots\alpha_{2n})$ とすれば

$$(2n-1)!!=\frac{(2n+1)!!}{2n+1}=\frac{(2n)!}{2^nn!}\qquad((2n)!!=2n(2n-2)\cdots2=2^nn!)$$

- $\mathrm{tr}\gamma^5=0$

  巡回性と $\gamma^0\gamma^5=-\gamma^5\gamma^0$ から

$$\mathrm{tr}\gamma^5=\mathrm{tr}[\gamma^5\gamma^0\gamma^0]=\mathrm{tr}[\gamma^0\gamma^5\gamma^0]=-\mathrm{tr}[\gamma^5\gamma^0\gamma^0]=-\mathrm{tr}[\gamma^5]=0$$

- $\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu]=0$

  $\mu=\nu$ では

$$\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\mu]=\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\mu]=2g^{\mu\mu}\mathrm{tr}\gamma^5=0$$

$\mu\neq\nu$ では、$\alpha\neq\mu,\nu$ として

$$
\begin{aligned}
\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu] &= \mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu(\gamma^\alpha)^{-1}\gamma^\alpha] \\
&= \mathrm{tr}[\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu(\gamma^\alpha)^{-1}] \\
&= -\mathrm{tr}[\gamma^5\gamma^\alpha\gamma^\mu\gamma^\nu(\gamma^\alpha)^{-1}] \\
&= -\mathrm{tr}[\gamma^5(2g^{\alpha\mu}-\gamma^\mu\gamma^\alpha)\gamma^\nu(\gamma^\alpha)^{-1}] \quad (g^{\alpha\mu}=0,\ \alpha\neq\mu) \\
&= (-1)^2\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\alpha\gamma^\nu(\gamma^\alpha)^{-1}] \\
&= (-1)^3\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha(\gamma^\alpha)^{-1}] \\
&= -\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu] \\
&= 0
\end{aligned}
$$

- $\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = -4i\epsilon^{\mu\nu\rho\sigma}$

  $\mu=\nu$ とすると

$$
\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\mu\gamma^\alpha\gamma^\beta] = g^{\mu\mu}\mathrm{tr}[\gamma^5\gamma^\alpha\gamma^\beta] = 0
$$

$\mu=\alpha$ では

$$
\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\beta] = -\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\mu\gamma^\nu\gamma^\beta] = 0
$$

$\mu=\beta$ では

$$
\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\mu] = \mathrm{tr}[\gamma^\mu\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha] = -\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\mu\gamma^\nu\gamma^\alpha] = 0
$$

なので、μ,ν,α,β のうち 2 個が同じなら 0 です。このため、$\gamma^0,\gamma^1,\gamma^2,\gamma^3$ を 1 個ずつ含む場合だけが 0 ではないです。よって

$$
\mathrm{tr}[\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3] = -i\mathrm{tr}[\gamma^5\gamma^5] = -4i
$$

ガンマ行列はそれぞれ交換するので、$0,1,2,3$ の並びに対して

$$
\mathrm{tr}[\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3] = -4i\ ,\ \mathrm{tr}[\gamma^5\gamma^0\gamma^1\gamma^3\gamma^2] = +4i\ ,\ \mathrm{tr}[\gamma^5\gamma^0\gamma^2\gamma^1\gamma^3] = +4i\ ,\ \ldots
$$

入れ替えで符号が反転し、2 個同じなら消えることはレヴィ・チビタ記号 $\epsilon^{\mu\nu\alpha\beta}$ に対応するので

$$\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = -4i\epsilon^{\mu\nu\alpha\beta}$$

添え字が下付きのときは、$\gamma_0 = \gamma^0, \gamma_i = (\gamma^i)^\dagger = -\gamma^i$ から

$$\mathrm{tr}[\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3] = -\mathrm{tr}[\gamma^5\gamma_0\gamma_1\gamma_2\gamma_3]$$

なので

$$\mathrm{tr}[\gamma^5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta] = +4i\epsilon^{\mu\nu\alpha\beta} = -4i\epsilon_{\mu\nu\alpha\beta}$$

このため、内積の表記

$$a_\mu a_\nu a_\alpha a_\beta \mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = a^\mu a^\nu a^\alpha a^\beta \mathrm{tr}[\gamma^5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta]$$

に対して

$$A_\mu B_\nu C_\alpha D_\beta \mathrm{tr}[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta] = -4i\epsilon^{\mu\nu\alpha\beta}A_\mu B_\nu C_\alpha D_\beta$$
$$A^\mu B^\nu C^\alpha D^\beta \mathrm{tr}[\gamma^5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta] = -4i\epsilon_{\mu\nu\alpha\beta}A^\mu B^\nu C^\alpha D^\beta$$

ベクトルとの内積を取った形で知っておくと便利なものとして

- $A_\alpha B_\beta \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta] = 4(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B))$
  $\mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta]$ を使って

$$\begin{aligned} A_\alpha B_\beta \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta] &= 4A_\alpha B_\beta(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\alpha\nu} - g^{\mu\nu}g^{\alpha\beta}) \\ &= 4(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B)) \end{aligned}$$

- $A_\alpha B_\beta \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta] = 8((A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B)) - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta)$
  γ^5 はガンマ行列と反交換するので

$$\begin{aligned} \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta] &= \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu(1-\gamma^5)(1-\gamma^5)\gamma^\beta] \\ &= 2\mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta] \\ &= 2\mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta] + 2\mathrm{tr}[\gamma^5\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta] \\ &= 8(g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\alpha}g^{\nu\beta}) - 8i\epsilon^{\mu\alpha\nu\beta} \end{aligned}$$

内積を取れば

$$A_\alpha B_\beta \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta] = 8(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B)) - 8i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta$$

- $A_\alpha B_\beta C^\rho D^\lambda \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta]\mathrm{tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\lambda] = 32((A\cdot C)(B\cdot D)+(A\cdot D)(B\cdot C))$

$$\begin{aligned}
&A_\alpha B_\beta C^\rho D^\lambda \mathrm{tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta]\mathrm{tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\lambda] \\
&\quad = 16(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B))(C_\mu D_\nu + C_\nu D_\mu - g_{\mu\nu}(C\cdot D)) \\
&\quad = 16\big((A\cdot C)(B\cdot D)+(A\cdot D)(B\cdot C)-(A\cdot B)(C\cdot D) \\
&\qquad\quad + (A\cdot D)(B\cdot C)+(A\cdot C)(B\cdot D)-(A\cdot B)(C\cdot D) \\
&\qquad\quad - (A\cdot B)(C\cdot D)-(A\cdot B)(C\cdot D)+4(A\cdot B)(C\cdot D)\big) \\
&\quad = 32((A\cdot C)(B\cdot D)+(A\cdot D)(B\cdot C))
\end{aligned}$$

- $A_\alpha B_\beta C^\rho D^\lambda \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta]\mathrm{tr}[\gamma_\mu(1-\gamma^5)\gamma_\rho\gamma_\nu(1-\gamma^5)\gamma_\lambda] = 256(A\cdot C)(B\cdot D)$

$$\begin{aligned}
&A_\alpha B_\beta C^\rho D^\lambda \mathrm{tr}[\gamma^\mu(1-\gamma^5)\gamma^\alpha\gamma^\nu(1-\gamma^5)\gamma^\beta]\mathrm{tr}[\gamma_\mu(1-\gamma^5)\gamma_\rho\gamma_\nu(1-\gamma^5)\gamma_\lambda] \\
&\quad = 64(A^\mu B^\nu + A^\nu B^\mu - g^{\mu\nu}(A\cdot B) - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta) \\
&\qquad \times (C_\mu D_\nu + C_\nu D_\mu - g_{\mu\nu}(C\cdot D) - i\epsilon_{\mu\rho\nu\lambda}C^\rho D^\lambda) \\
&\quad = 64\big((A\cdot C)(B\cdot D)+(A\cdot D)(B\cdot C)-(A\cdot B)(C\cdot D) - i\epsilon_{\mu\rho\nu\lambda}A^\mu B^\nu C^\rho D^\lambda \\
&\qquad\quad + (A\cdot D)(B\cdot C)+(A\cdot C)(B\cdot D)-(A\cdot B)(C\cdot D) - i\epsilon_{\mu\rho\nu\lambda}A^\nu B^\mu C^\rho D^\lambda \\
&\qquad\quad - (A\cdot B)(C\cdot D)-(A\cdot B)(C\cdot D)+4(A\cdot B)(C\cdot D) + i\epsilon_{\mu\rho\nu\lambda}g^{\mu\nu}(A\cdot B)C^\rho D^\lambda \\
&\qquad\quad - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta C_\mu D_\nu - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta C_\nu D_\mu + i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta g_{\mu\nu}(C\cdot D) \\
&\qquad\quad - \epsilon^{\mu\alpha\nu\beta}\epsilon_{\mu\rho\nu\lambda}A_\alpha B_\beta C^\rho D^\lambda\big) \\
&\quad = 64(2(A\cdot C)(B\cdot D)+2(A\cdot D)(B\cdot C) \\
&\qquad\quad - i\epsilon_{\mu\rho\nu\lambda}A^\mu B^\nu C^\rho D^\lambda - i\epsilon_{\mu\rho\nu\lambda}A^\nu B^\mu C^\rho D^\lambda \\
&\qquad\quad - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta C_\mu D_\nu - i\epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta C_\nu D_\mu + i\epsilon^{\mu\alpha\nu\beta}g_{\mu\nu}A_\alpha B_\beta(C\cdot D) \\
&\qquad\quad - \epsilon^{\mu\alpha\nu\beta}\epsilon_{\mu\rho\nu\lambda}A_\alpha B_\beta C^\rho D^\lambda)
\end{aligned}$$

$g^{\mu\nu}$ は $\mu=\nu$ のときに ±1 なので $\epsilon_{\mu\rho\nu\lambda}g^{\mu\nu}$ は 0 です。レヴィ・チビタ記号を 1 個含む項は

$$\begin{aligned}
&\epsilon_{\mu\rho\nu\lambda}A^\mu B^\nu C^\rho D^\lambda + \epsilon_{\mu\rho\nu\lambda}A^\nu B^\mu C^\rho D^\lambda + \epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta C_\mu D_\nu + \epsilon^{\mu\alpha\nu\beta}A_\alpha B_\beta C_\nu D_\mu \\
&\quad = -\epsilon_{\mu\nu\rho\lambda}A^\mu B^\nu C^\rho D^\lambda + \epsilon_{\nu\mu\rho\lambda}A^\nu B^\mu C^\rho D^\lambda - \epsilon^{\alpha\beta\mu\nu}A_\alpha B_\beta C_\mu D_\nu + \epsilon^{\alpha\beta\nu\mu}A_\alpha B_\beta C_\nu D_\mu \\
&\quad = 0
\end{aligned}$$

最後に示しているように

$$\epsilon^{\mu\alpha\nu\beta}\epsilon_{\mu\rho\nu\lambda}=\epsilon^{\alpha\beta\mu\nu}\epsilon_{\rho\lambda\mu\nu}=2(\delta^{\alpha}_{\lambda}\delta^{\beta}_{\rho}-\delta^{\alpha}_{\rho}\delta^{\beta}_{\lambda}) \tag{1}$$

から、レヴィ・チビタ記号を 2 個含む項は

$$A_\alpha B_\beta C^\rho D^\lambda(\delta^{\alpha}_{\lambda}\delta^{\beta}_{\rho}-\delta^{\alpha}_{\rho}\delta^{\beta}_{\lambda})=A_\lambda B_\rho C^\rho D^\lambda-A_\rho B_\lambda C^\rho D^\lambda=(A\cdot D)(B\cdot C)-(A\cdot C)(B\cdot D)$$

よって

$$\begin{aligned}&64(2(A\cdot C)(B\cdot D)+2(A\cdot D)(B\cdot C)-\epsilon^{\mu\alpha\nu\beta}\epsilon_{\mu\rho\nu\lambda}A_\alpha B_\beta C^\rho D^\lambda)\\&=64(2(A\cdot C)(B\cdot D)+2(A\cdot D)(B\cdot C)-2(A\cdot D)(B\cdot C)+2(A\cdot C)(B\cdot D))\\&=256(A\cdot C)(B\cdot D)\end{aligned}$$

最後に (1) を示します。$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\lambda\alpha\beta}$ は

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\lambda\alpha\beta}=\epsilon^{\mu\nu01}\epsilon_{\rho\lambda01}+\cdots+\epsilon^{\mu\nu10}\epsilon_{\rho\lambda10}+\cdots+\epsilon^{\mu\nu20}\epsilon_{\rho\lambda20}+\cdots$$

このとき 0 にならないのは

$$\epsilon^{\mu\nu01}\epsilon_{\rho\lambda01}\ \Rightarrow\ \epsilon^{2301}\epsilon_{2301}\ ,\ \epsilon^{2301}\epsilon_{3201}$$

のように、$\mu=\rho,\nu=\lambda$ か $\mu=\lambda,\nu=\rho$ のどちらかだけです。そして、α,β は μ,ν と同じでは 0 になるので、$\mu=\rho,\nu=\lambda$ では、$\epsilon^{0123}\epsilon_{0123}=\epsilon^{0132}\epsilon_{0132}=\cdots=-1$ から

$$\epsilon^{abcd}\epsilon_{abcd}+\epsilon^{abdc}\epsilon_{abdc}=-2$$

例えば、$\mu=1,\nu=2$ なら

$$\epsilon^{12\alpha\beta}\epsilon_{12\alpha\beta}=\epsilon^{1203}\epsilon_{1203}+\epsilon^{1230}\epsilon_{1230}=-2$$

となるからです。$\mu=\lambda,\nu=\rho$ では符号が反転し

$$\epsilon^{abcd}\epsilon_{bacd}+\epsilon^{abdc}\epsilon_{badc}=+2$$

よって、クロネッカーデルタで $\mu=\rho,\nu=\lambda$ と $\mu=\lambda,\nu=\rho$ で分けて

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\lambda\alpha\beta}=2(\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho}-\delta^{\mu}_{\rho}\delta^{\nu}_{\lambda})$$